Chapter VII : Key Results

Fermions

- Find $\{n_i\}_{mp.}$ that maximizes $W_{FD}(\{n_r\}) = \prod_i \frac{g_i!}{n_i!(g_i-n_i)!}$ under the constraint

$$E = \sum_i n_i \varepsilon_i = \text{constant}$$
$$N = \sum_i n_i = \text{constrant}$$

- $$\frac{n_i}{g_i} = \frac{1}{e^{\alpha} e^{\beta \varepsilon_i} + 1} = \frac{1}{e^{\beta(\varepsilon_i - \mu)} + 1}$$

$$\boxed{f_{FD}(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)} + 1} \quad \text{Fermi-Dirac distribution}}$$

Meaning: Number of fermions (particles) per single-particle state at energy ε

Bosons

- Find $\{n_i\}_{mp}$ that maximizes $W_{BE}(\{n_i\}) = \prod_i \frac{(g_i+n_i)!}{g_i!\, n_i!}$ under the constraint

$$E = \sum_i n_i \varepsilon_i = \text{constant}$$
$$N = \sum_i n_i = \text{constant}$$

- $$\frac{n_i}{g_i} = \frac{1}{e^{\alpha} e^{\beta \varepsilon_i} - 1} = \frac{1}{e^{\beta(\varepsilon_i - \mu)} - 1}$$

$$\boxed{f_{BE}(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)} - 1} \quad \text{Bose-Einstein distribution}}$$

Meaning: Number of bosons (particles) per single-particle state of energy ε

Using the constraint:

$$\boxed{\begin{aligned} N &= \sum_i g_i \frac{1}{e^{\beta(\varepsilon_i-\mu)}+1} \\ E &= \sum_i g_i \frac{\varepsilon_i}{e^{\beta(\varepsilon_i-\mu)}+1} \end{aligned}}$$

Look at the equation

- Eq. for μ as a function of T

Using the constraint:

$$\boxed{\begin{aligned} N &= \sum_i g_i \frac{1}{e^{\beta(\varepsilon_i-\mu)}-1} \\ E &= \sum_i g_i \frac{\varepsilon_i}{e^{\beta(\varepsilon_i-\mu)}-1} \end{aligned}}$$

- Single-particle states are closed spaced in energy

$\Rightarrow$ Continuous description $g(\varepsilon)d\varepsilon$ = number of single particle states with energy in the interval ε to $\varepsilon + d\varepsilon$

$$N = \int \frac{g(\varepsilon)}{e^{\beta(\varepsilon-\mu)}+1} d\varepsilon$$
$$E = \int \frac{g(\varepsilon)\, \varepsilon}{e^{\beta(\varepsilon-\mu)}+1} d\varepsilon$$

- Eq. for μ as a function of T

$$N = \int \frac{g(\varepsilon)}{e^{\beta(\varepsilon-\mu)}-1} d\varepsilon$$
$$E = \int \frac{g(\varepsilon)\, \varepsilon}{e^{\beta(\varepsilon-\mu)}-1} d\varepsilon$$

What's next?

$$N = \sum_i g_i \frac{1}{e^{\beta(\varepsilon_i-\mu)} \pm 1}$$

$$E = \sum_i g_i \frac{\varepsilon_i}{e^{\beta(\varepsilon_i-\mu)} \pm 1}$$

Look into $\underbrace{g_i}$

number of single-particle states
at some given energy

OR more generally (practically)

$g(\varepsilon)d\varepsilon$ = number of single-particle states
in the energy interval ε to $\varepsilon + d\varepsilon$
(called "density of single-particle states"
OR "density of states")

$g(\varepsilon)$ is NOT a Stat. Mech. problem. It is more a QM problem.

F. Discussions

- Here, we applied the Lagrange multipliers method as a quick and dirty way to obtain the Fermi-Dirac and Bose-Einstein distribution. Later, we will obtain these distributions again by invoking the Gibbs distribution and the Grand Canonical Ensemble, which are more convenient in treating quantum gases.
- Formally, the multiplers α and β should be determined by the constraints. Here, we claimed the established results of $\alpha = -\frac{\mu}{kT}$ and $\beta = \frac{1}{kT}$.
- The analysis also shows when quantum gases become classical ideal gas. We need $\frac{n_i}{g_i} \ll 1$, which amounts to separation $\gg \lambda_{thermal}$.
- We will treat ideal fermi gas and ideal bose gas in later chapters.
- The FD and BE distributions can also be obtained by minimizing F.

Summary

students should be able to:

- carry out the process of minimizing or maximizing a quantity under some constraints, using the method of Lagrangian multiplers
- obtain the Fermi-Dirac and Bose-Einstein distributions
- understand what $f_{FD}(\varepsilon)$ and $f_{BE}(\varepsilon)$ mean
- understand that when $n_i \ll g_i$, the restrictions imposed by QM on single-particle state occupancy become un-important.
- use the Maxwell-Bolzmann distribution (after some practice) in classical statistical mechanics.

Refs:

The way (quick and dirty) we get the distributions is discussed in more elementary books.

- Trerena: Ch3,4,5,6 ; Guénault: Ch.2,4,5
- All undergraduate books in Chinese.

  e.g. 苏汝铿,「统计物理学」(第四章)

  龚昌德,「热力学与统计物理学」(第七章)